# On the $3 \times 3$ magic square constructed with nine distinct square numbers

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**Abstract**

A proof that there is no $3 \times 3$ magic square constructed with nine distinct square numbers is given.

## 1 Introduction

In 1984 Martin Labar [1] formulated the problem: Can a $3 \times 3$ magic square be constructed with nine distinct square numbers? The problem is found in the second edition of Guy's *Unsolved Problems in Number Theory* [2] and became famous when Martin Gardner republished it in 1996 [3].

## 2 The proof

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $d$ | $\varepsilon$ | $f$ |
| $g$ | $h$ | $i$ |

Figure 1

Let be the square given in Figure 1 such that $a, b, c, d, \varepsilon, f, g, h, i \in \mathbf{N}$ and

$$a + b + c = x \tag{1}$$
$$d + \varepsilon + f = x \tag{2}$$
$$g + h + i = x \tag{3}$$
$$a + d + g = x \tag{4}$$
$$b + \varepsilon + h = x \tag{5}$$
$$c + f + i = x \tag{6}$$
$$a + \varepsilon + i = x \tag{7}$$
$$c + \varepsilon + g = x \tag{8}$$

The equations (1), (2), (3), (5), (6) (4), (7) and (8) can be rewritten as

$$a = x - b - c \tag{9}$$

$$d = x - \varepsilon - f \tag{10}$$

$$g = x - \varepsilon - i \tag{11}$$

$$b = x - \varepsilon - h \tag{12}$$

$$c = x - f - i \tag{13}$$

$$a + d + g - x = 0 \tag{14}$$

$$a + \varepsilon + i - x = 0 \tag{15}$$

$$c + \varepsilon + g - x = 0 \tag{16}$$

Substituting sequentially $a$, $d$, $g$, $b$ and $c$ given by (10) to (13) into the equations (14) to (16) we obtain, respectively,

$$0 = 0 \tag{17}$$

$$2\varepsilon + f + h + 2i - 2x = 0 \tag{18}$$

$$\varepsilon - f - h - 2i + x = 0 \tag{19}$$

Summing (18) and (19) we find

$$\varepsilon = \frac{x}{3} \tag{20}$$

The set of equations (1) to (8) can be put in the form

$$a = \varepsilon + \Delta_1 \tag{21}$$

$$b = \varepsilon - (\Delta_1 + \Delta_2) \tag{22}$$

$$c = \varepsilon + \Delta_2 \tag{23}$$

$$d = \varepsilon - (\Delta_1 - \Delta_2) \tag{24}$$

$$f = \varepsilon + (\Delta_1 - \Delta_2) \tag{25}$$

$$g = \varepsilon - \Delta_2 \tag{26}$$

$$h = \varepsilon + (\Delta_1 + \Delta_2) \tag{27}$$

$$i = \varepsilon - \Delta_1 \tag{28}$$

Let us notice that $\Delta_1 \neq 0$, $\Delta_2 \neq 0$ and $\Delta_1 \neq \Delta_2$ to obtain the magic square constructed with nine distinct square numbers.

Let us consider that

$$\eta_n = (n+1)^2 - n^2 \tag{29}$$

$$\eta_{n+1} = \eta_n + 2 \tag{30}$$

where $n \in \mathbf{N}$ and $n > 0$. The Figure 2 was obtained using the equations (29) and (30).

$$\begin{array}{ccccccccccccc}
1^2 & & 2^2 & & 3^2 & & 4^2 & & 5^2 & & 6^2 & & ... \\
1 & & 4 & & 9 & & 16 & & 25 & & 36 & & ... \\
 & 3 & & 5 & & 7 & & 9 & & 11 & & & ...
\end{array}$$

Figure 2

Let us assume that there exist $0 < m < e$ and $e < n$ such that

$$n^2 + m^2 = 2e^2 \tag{31}$$

The values of $n^2$ and $m^2$ are

$$n^2 = e^2 + \sum_{k=e}^{n-1} \eta_k \tag{32}$$

and

$$m^2 = e^2 - \sum_{k=m}^{e-1} \eta_k \tag{33}$$

or

$$n^2 = e^2 + \frac{\eta_e + \eta_{n-1}}{2}((n-1) - (e-1)) \tag{34}$$

and

$$m^2 = e^2 - \frac{\eta_m + \eta_{e-1}}{2}((e-1) - (m-1)) \tag{35}$$

Considering $a = n^2$, $i = m^2$ and $\varepsilon = e^2$ we have

$$\frac{\eta_e + \eta_{n-1}}{2}((n-1) - (e-1)) - \frac{\eta_m + \eta_{e-1}}{2}((e-1) - (m-1)) = 0 \tag{36}$$

or

$$\begin{aligned} &(-e+n)(-2e^2 + 2(1+e)^2 + 2(-1-e+n)) - \\ &\quad ((-2 - 2e^2 + 2(1+e)^2 - 2(e-m))(e-m) = 0 \end{aligned} \tag{37}$$

Let us assume that there exist $w$ and $z$ such that

$$c = (n+w)^2 \tag{38}$$

and

$$g = (m-z)^2 \tag{39}$$

where $w$ and $z$ are positive integers. In this case we have

$$\begin{aligned} &(-e+n+w)(-2e^2 + 2(1+e)^2 + 2(-1-e+n+w)) - \\ &\quad ((-2 - 2e^2 + 2(1+e)^2 - 2(e-m+z))(e-m+z) = 0 \end{aligned} \tag{40}$$

Subtracting (37) from (40) we obtain

$$2nw + w^2 + (-2m + z) = 0 \tag{41}$$

Solving (41) for $z$ we find

$$z_1 = m + \sqrt{m^2 - 2nw - w^2} \tag{42}$$

and

$$z_2 = m - \sqrt{m^2 - 2nw - w^2} \tag{43}$$

The root $z_1$ implies that $m - z$ is not a positive integer, however $m - z$ must be a positive integer. Therefore $z = z_2$.

We have assumed

$$2e^2 = m^2 + n^2 \tag{44}$$

and

$$2e^2 = (m-z)^2 + (n+w)^2 \tag{45}$$

Subtracting (44) from (45) we obtain

$$(m-z)^2 + (n+w)^2 - (m^2 + n^2) = 0 \tag{46}$$

Substituting (43) into (46) we have

$$n^2 - 2nw - w^2 - (n+w)^2 = 0 \tag{47}$$

Solving (47) for $w$ we find

$$w_1 = 0 \tag{48}$$

and

$$w_2 = -2n \tag{49}$$

The root $w_2$ implies in $n+w$ negative, however $n+w$ must be a positive integer. Therefore $w = 0$. Since that $w = 0$ the condition $\Delta_1 \neq \Delta_2$ is not satisfied.There is no magic square constructed with nine distinct square numbers.

# References


[1] Martin Labar, *Problem 270*, College Math. J. 15, pp. 69, 1984.

[2] Richard Guy, *Unsolved Problems in NumberTheory*, 2nd edition, Springer-Verlag, New York, Problem D15, pp. 170-171, 1994.

[3] Martin Gardner, *The magic of* $3 \times 3$, Quantum, pp. 24-26,1996.